Disjoint Stable Matchings in Linear Time

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The Marriage Model

In the marriage model, we are given with a bipartite graph $G = (A \cup B, E)$, and for each $v \in A \cup B$ a strict ordering $\succ_v$ of its neighbours - given in it's preference list.

A: Set of menB: Set of women

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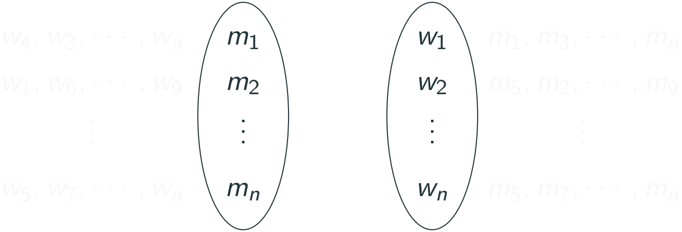

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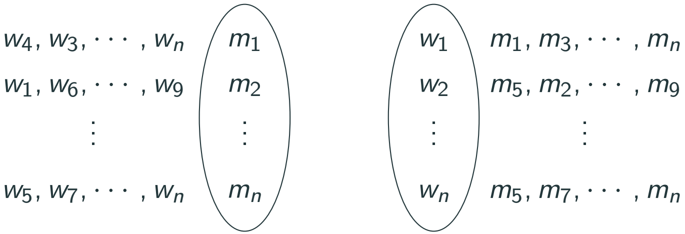


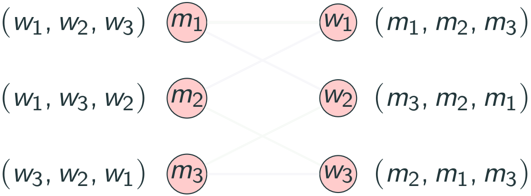
A: Set of men B: Set of women

Stability

A matching M is said to be stable if there is no edge $(m, w) \in E \backslash M$ such that:

$$w \succ_m M(m) \quad \text{and} \quad m \succ_w M(w)$$

That is, m and w prefer each other over their respective partners in M.



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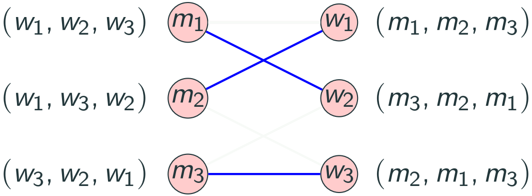

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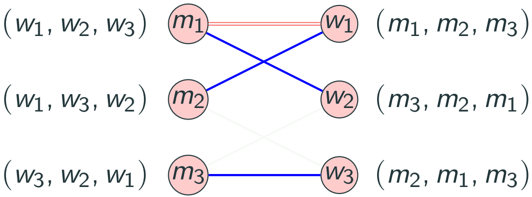

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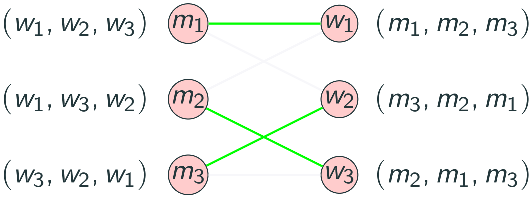

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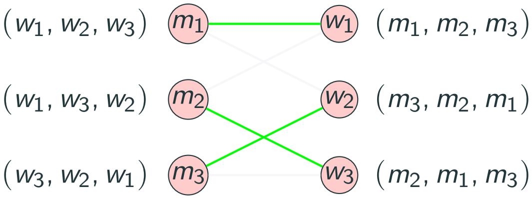

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Gale and Shapley Algorithm

Unmatched men propose. Women accept or reject based on their preference list.

Key Results:

1. All possible execution of the Gale-Shapley algorithm yields the same result.
2. It results in *"Man-optimal"* stable matching.

   Man-optimal: Every man is matched with his most favored partner among all stable partners.
3. Reversing roles, i.e, women proposing, results in *"Woman-optimal"* stable matching.

   Woman-optimal: Every woman is matched with her most favored partner among all stable partners.
4. The man-optimal stable matching is *woman-pessimal*, and vice-versa.

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Extended Gale-Shapley Algorithm

Extended Gale-Shapley(EGS) algorithm is very similar to the Gale-Shapley algorithm except - EGS modifies the input preference list.

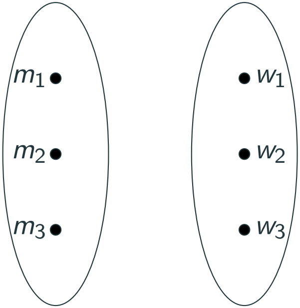
Run of Extended GS Algorithm

m_1 :	w_2	w_3	w_1
m_2 :	w_3	w_1	w_2
m_3 :	w_2	w_1	w_3

Men's Preference

w_1 :	m_1	m_2	m_3
w_2 :	m_2	m_1	m_3
w_3 :	m_3	m_2	m_1

Women's Preference



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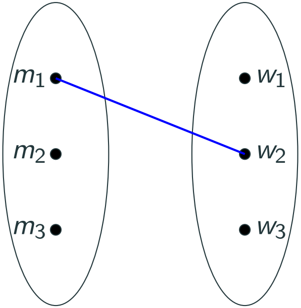

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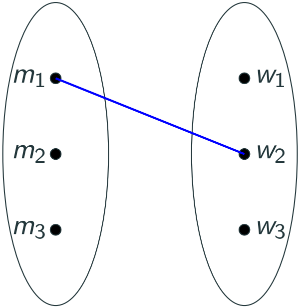
m_2 :	w_3	w_1	w_2
m_3 :		w_1	w_3

Men's Preference

w_1 :	m_1	m_2	m_3
w_2 :	m_2	m_1	
w_3 :	m_3	m_2	m_1

Women's Preference



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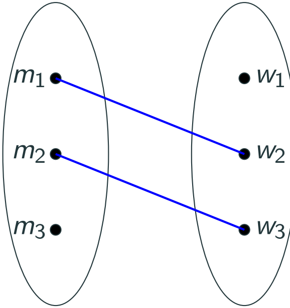

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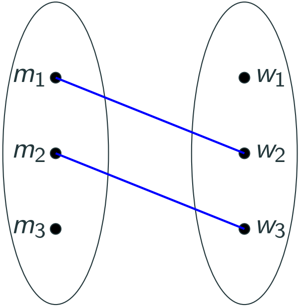

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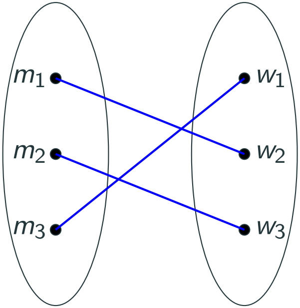

A person x is said to *prefer* a matching M to a matching M' if x prefers $p_M(x)$ to $p_{M'}(x)$.

Domination

A stable matching M is said to *dominate* a stable matching M', written $M \preceq M'$, if every man has at least as good a partner in M as he has in M' i.e., every man either prefers M to M' or is indifferent between them. *M strictly dominates M'* ($M \prec M'$) if $M \preceq M'$ and $M \cap M' = \varnothing$.

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The Lattice Structure

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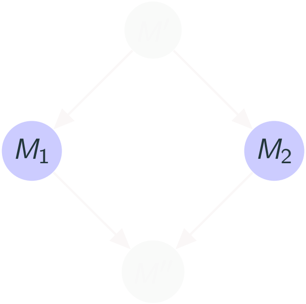
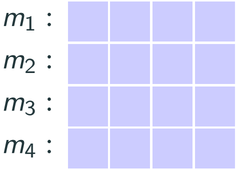
Men's Preference

Meet and Join





Men's Preference

$$M' = \{(m, w) \mid w = best(p_{M_1}(m), p_{M_2}(m))\}$$

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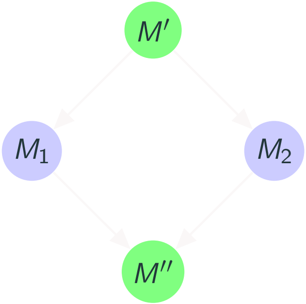

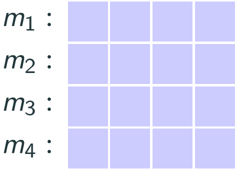

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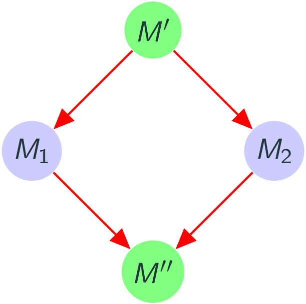

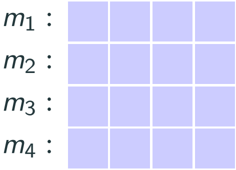

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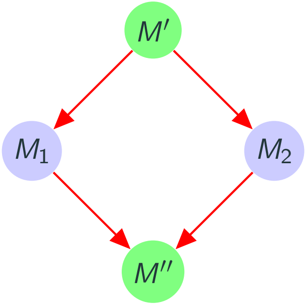

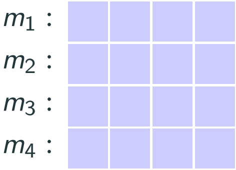

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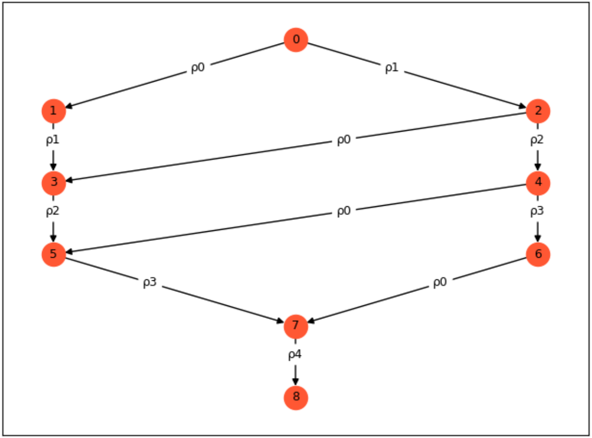
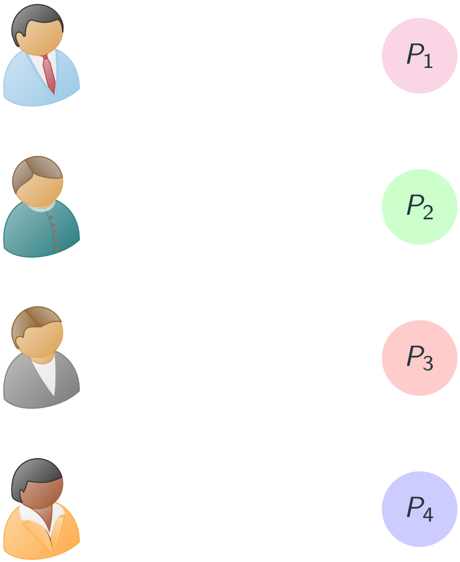
The Lattice Structure

Set of all stable matchings form a distributive lattice under the *Domination* domination.



Disjoint Stable Matchings

Why do we need them?



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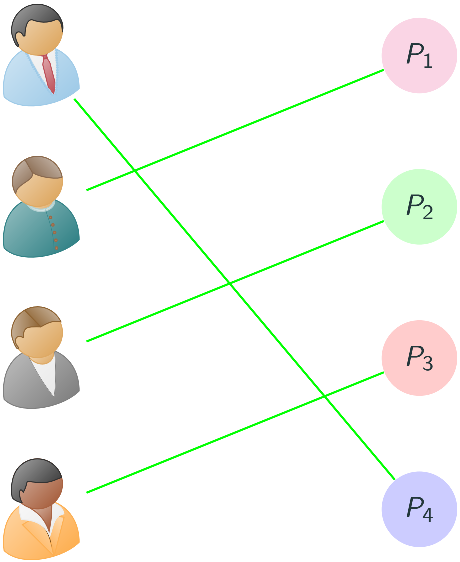

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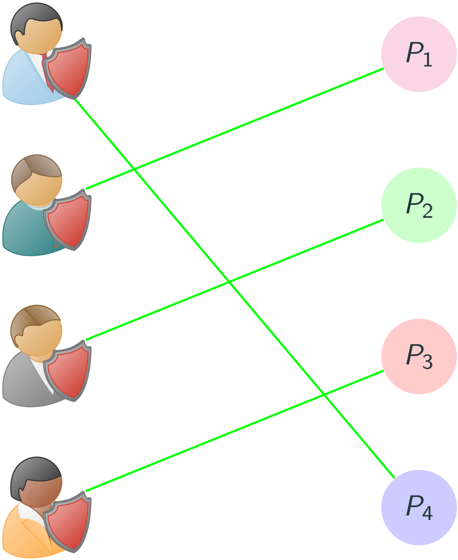

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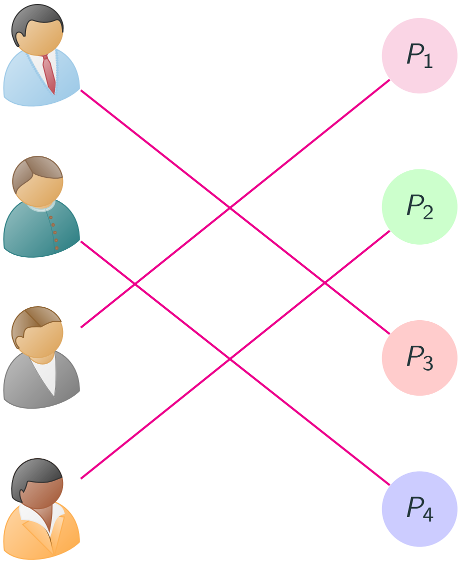

Problem Statement

For a given marriage instance, find a largest set S of disjoint stable matchings.

Existence of Disjoint Stable Matchings

Does there exist a marriage matching instances with disjoint stable matchings?

$m_1 : w_1, w_2, w_3$	$w_1 : m_2, m_3, m_1$
$m_2 : w_2, w_3, w_1$	$w_2 : m_3, m_1, m_2$
$m_3 : w_3, w_1, w_2$	$w_3 : m_1, m_2, m_3$

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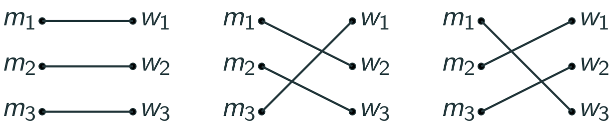

Necessary Condition

If the man-optimal and the woman-optimal stable matchings share a common edge (m, w), then (m, w) is in every stable matching.

This is because w is both the best stable partner and the worst stable partner of m.

So, to have disjoint stable matchings, man-optimal and woman-optimal matchings must be disjoint.

Algorithm: Disjoint Stable Matchings

- Input: Marriage instance G, Empty set S.
- $X \leftarrow \textsc{Extended}\mathrm{GS}(G)$
- While $X \cap M_Z = \varnothing$
  - $S \leftarrow S \cup X$
  - Delete X from G
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Run of Disjoint GS Algorithm

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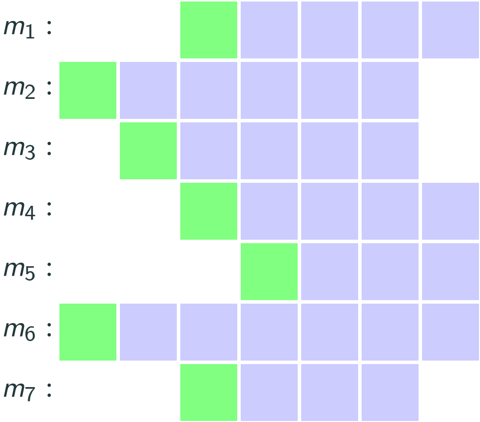
m_2 :							
m_3 :							
m_4 :							
m_5 :							
m_6 :							
m_7 :							

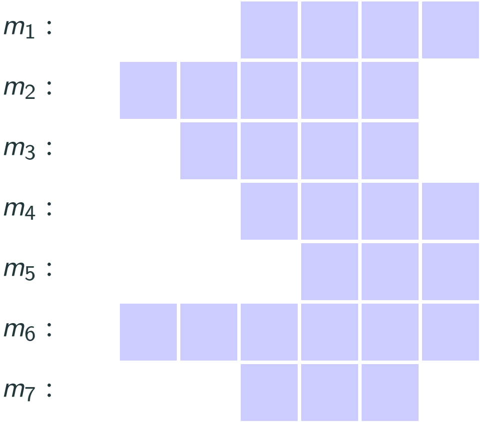
Men's preference list

Run of Disjoint GS Algorithm



Men's Preference

Run of Disjoint GS Algorithm



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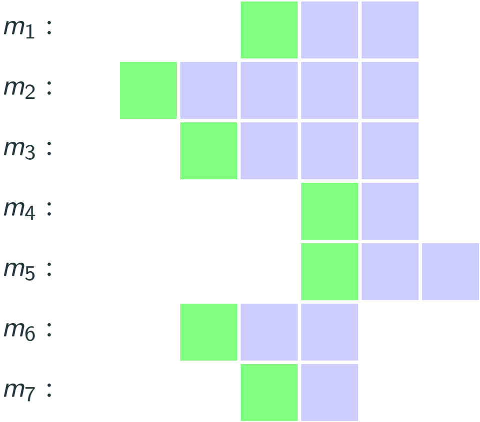

Men's Preference

Termination and Time Complexity

In every iteration, we delete at least one entry from the preference list. As the size of preference list is $2n^2$, the algorithm **terminates**.

For the same reason, the running time of the algorithm is $\mathbf{O}(\mathbf{n^2})$.

Disjoint Stable Matchings

Lemma 1

Each M_i in the set $S = \{M_0, M_1, \cdots, M_k\}$ is a perfect matching.

Note: It does not come freely from Extended GS!

It only guarantees one-one.

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Disjoint Stable Matchings

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All the matchings in the set S are stable matchings.

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If $M_0, M_1, \cdots, M_k$ are the matchings discovered by the algorithm in this order, then $M_0 \prec M_1 \prec \cdots \prec M_k$.

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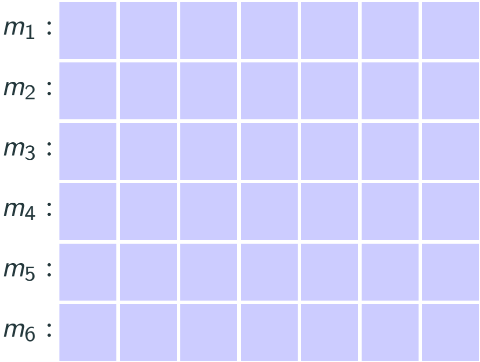
Lemma 4

In any arbitrary execution E of the algorithm, for any man m, $p_{M_i}(m)$ is the best stable partner of m when, for **every** *man, stable partners from $M_0, M_1, \cdots, M_{i-1}$ are disallowed.*



Lemma 5

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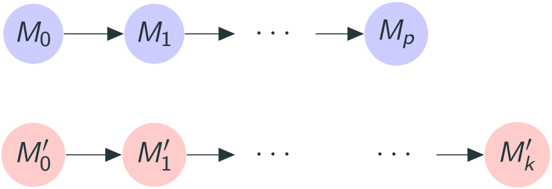

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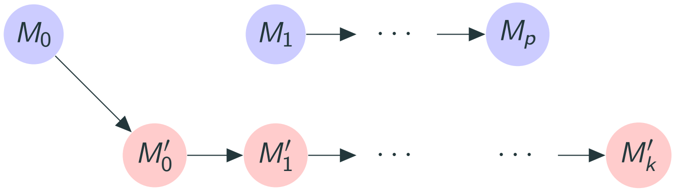

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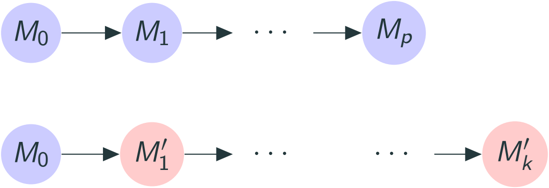

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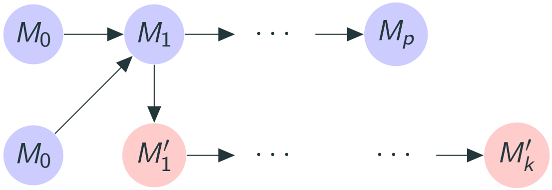

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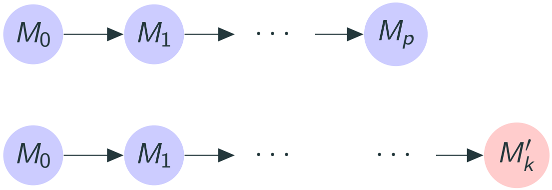

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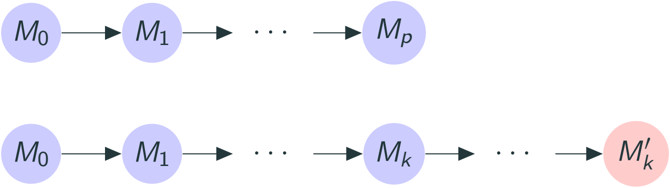

Disjoint Stable Matching

Theorem 6 (Teo, C.-P. and Sethuraman, J. (1998))

Let $S = \{M_1, M_2, \cdots, M_k\}$ *be a set of stable matchings for a particular stable matchings instance. For each man* m*, let* S_m *be the sorted multiset* $\{p_{M_1}(m), p_{M_2}(m), \cdots, p_{M_k}(m)\}$*, sorted according to the preference order of* m*. For every* $i \in \{1, 2, \cdots, k\}$ *let* $M_i' = \{(m, w) | m \in \mathcal{M}$ *and* w *is the* i^{th} *woman in* $S_m\}$*. Then for each* $i \in \{1, 2, \cdots, k\}$*,* M_i' *is a stable matching.*

Given stable matchings $M_1, M_2, \cdots, M_k$,

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Disjoint Chain

Corollary 7

Let $M_1, \ldots, M_k$ and $M'_1, \ldots, M'_k$ be as defined in 6. If $M_1, \ldots, M_k$ are pairwise disjoint, then $M'_1, \ldots, M'_k$ form a k-length chain of disjoint stable matchings.

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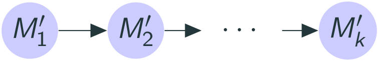

Maximum Size Set of Disjoint Stable Matchings

Theorem 8

For a given stable marriage instance, the algorithm gives the maximum size set of disjoint stable matchings.

Enumeration

- Our algorithm gives one of the largest sets of disjoint stable matching.
- Are there multiple solutions to the problem?

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(w_4, w_1, w_3, w_2) m_1 — w_1 (m_2, m_1, m_3, m_4)

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(w_1, w_3, w_2, w_4) m_3 — w_3 (m_4, m_2, m_3, m_1)

(w_1, w_4, w_2, w_3) m_4 — w_4 (m_3, m_4, m_2, m_1)

$S_1 = \{M_1, M_3\}$ and $S_2 = \{M_2, M_3\}$

(w_4, w_1, w_3, w_2) m_1 — w_1 (m_2, m_1, m_3, m_4)

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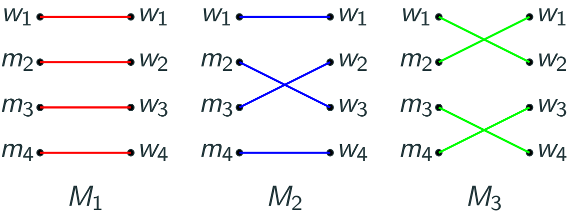

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(w_4, w_1, w_3, w_2) m_1 | w_1 (m_2, m_1, m_3, m_4)

(w_4, w_2, w_3, w_1) m_2 | w_2 (m_1, m_3, m_2, m_4)

(w_1, w_3, w_2, w_4) m_3 | w_3 (m_4, m_2, m_3, m_1)

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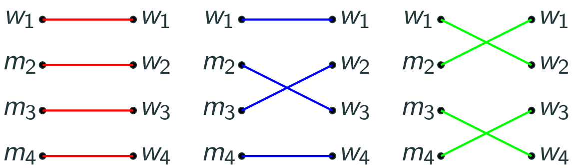

M_1 M_2 M_3

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Enumeration Algorithm

Enumerating all maximum length chains of disjoint stable matchings:

Given a marriage instance, we run our algorithm once in men-proposing settings and and once more in women-proposing setting to get the following chains of disjoint stable matchings.

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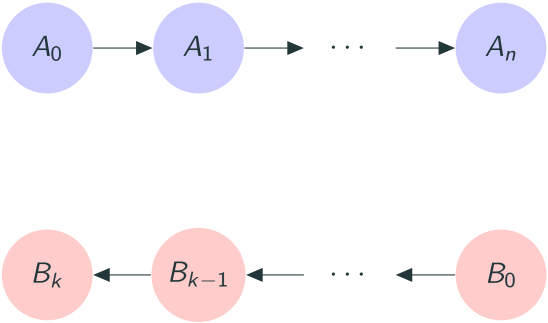

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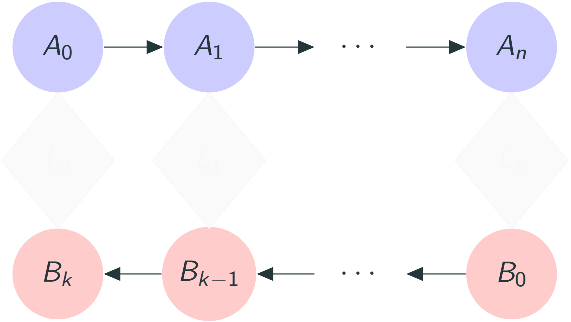

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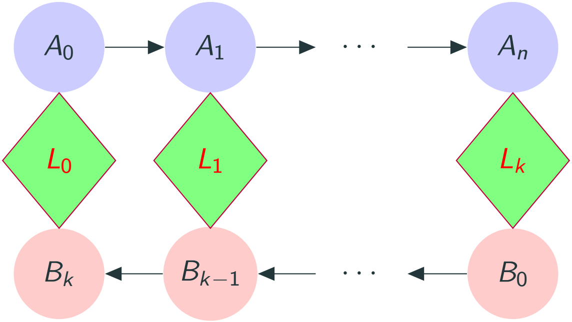

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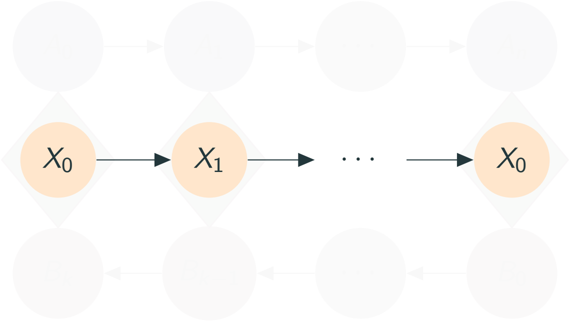
Let $X = \{X_0, \cdots X_k\}$ be a maximum-length chain of disjoint stable matchings i.e. $X_0 \prec X_1 \prec \cdots \prec X_k$. We note the following property of the matchings in X.

Lemma 9

For $0 \leq i \leq k$, $A_i \preceq X_i \preceq B_{k-i}$



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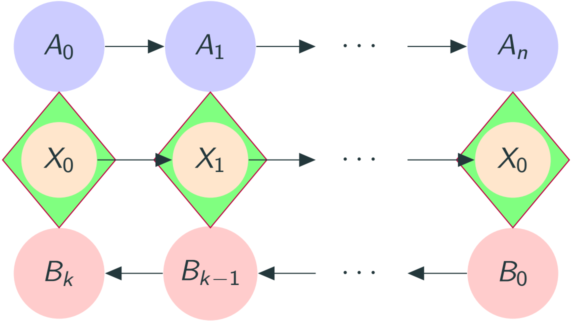

Enumeration Algorithm

With the help of lemma 9, we use *branching technique* to enumerate all possible max-length chains of disjoint stable matchings in *polynomial delay*.

Random Instance

We analyze the number of maximum-length chains of disjoint stable matchings in a random stable matchings instance with complete lists.

Lemma 10

The probability of the number of maximum size chains of disjoint stable matchings exceeding $(\frac{n}{\ln n})^{\ln n}$ *is at most* $O(\frac{(\ln n)^2}{n^2})$.

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Thank You!